Grade 7/8 Math Circles
Nov 20/21/22/23, 2023
Parabolas - Problem Set

1. For the following functions, fill in the table and plot the points from the table on a Cartesian-plane. Determine if the graph forms a parabola.

(a)

x	$y = -2x^2 + 3$	(x, y)
-3		
-2		
-1		
0		
1		
2		
3		

(b)

x	$y = x^4 - 1$	(x, y)
-3		
-2		
-1		
0		
1		
2		
3		

2. Answer the following questions about these these four functions **without** graphing.

(1) $y = -x^2 + 5$ (2) $y = 2x^2 - 1$ (3) $y = -4x^2$ (4) $y = 0.5x^2 + x$

(a) Determine whether the parabolas of the above functions open up or down.

(b) Order the functions from narrowest to widest.

3. Calculate the area under the parabolas from the lower bound to the upper bound on the x-axis. Hint: Start by graphing the function.

(a) $y = x^2 - 2x + 1$ from $x = -1$ to $x = 3$

(b) $y = 0.5x^2 + 2x + 2.5$ from $x = -5$ to $x = 1$

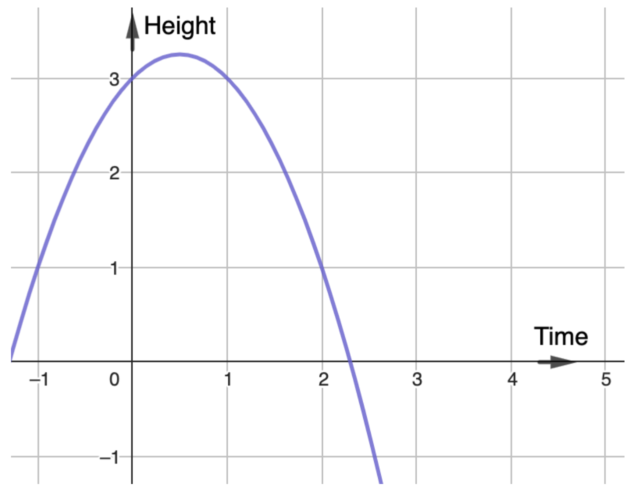
4. There's a pirate battle on the horizon! Captain Longbeard is ready to fire his cannon at the ship of the notorious Captain Redbeard! Longbeard knows that the cannonball must be in the air for exactly 4 seconds to make a direct hit. The cannon is always fired at time $t = 0$.

   (a) The arch that the cannonball follows can be described by the function $h = -t^2 + 4t + 3$, where h is the height above the sea in meters at time t in seconds.

   Here is the graph of the first cannonball's flight path.

   From what height was the first cannonball launched?

   Did the first shot stay in the air for the correct amount of time to hit Captain Redbeard's ship?

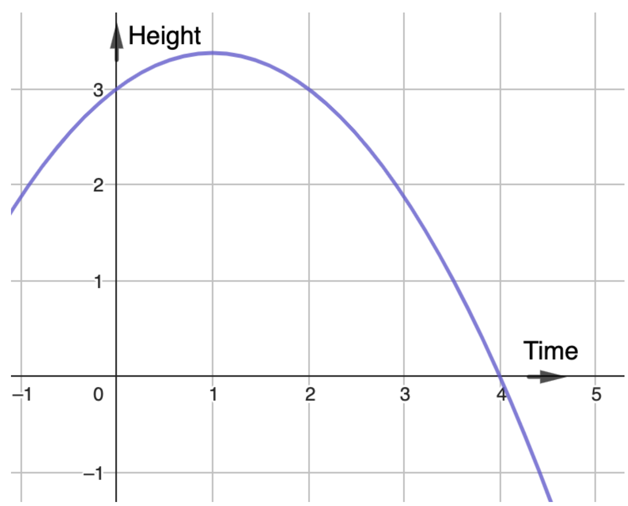


   (b) For the second shot, reset the timer so the shot occurs at time $t = 0$. The arch of this cannonball follows the function $h = -\frac{3}{8}x^2 + \frac{3}{4}x + 3$.

   Here is a graph of the second cannonball's light path.

   Did the second shot stay in the air for the correct amount of time to hit Captain Redbeard's ship?

   What is the maximum height of the second cannonball and when did it occur.

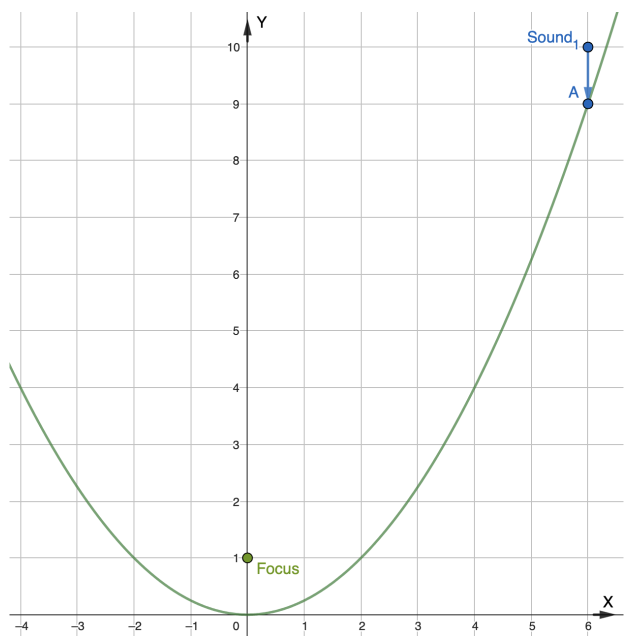


5. Sound amplifiers use a parabolic shape to gather sound waves by reflecting them to a microphone at the focus of the parabolic shield.

   But what if someone forgot to install the microphone? How would the sound waves behave after bouncing off the shield? Answer this by completing the paths of the sound wave on the right. Then make an observation about the path of the wave.

   HINT: No microphone is at the focus so the wave will continue straight passed the focus.

Image retrieved from Amazon



6. CHALLENGE QUESTION 1

   Recall that integrals like $\int_1^{2.5}(x^2+3x-4)\,dx$ can be used to calculate the area under functions. This one calculates the area under x^2+3x-4 from $x=1$ to $x=2.5$.

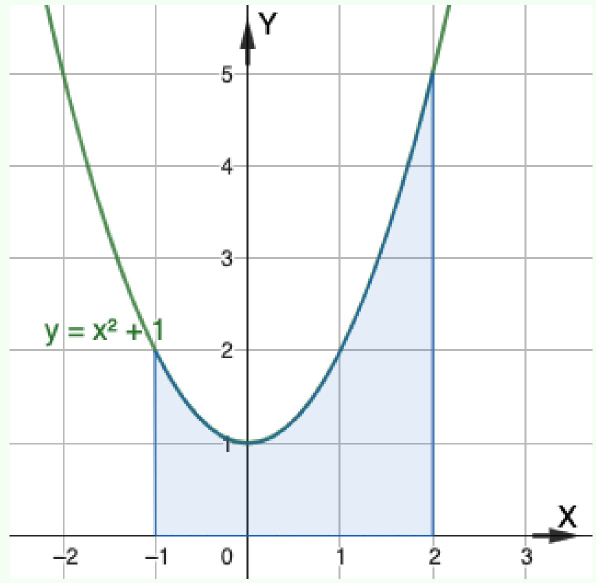
   Graph the function and shade in the area that the following integrals calculate.
   NOTE: You don't need to calculate the area, just shade the region. (But you can calculate the area using basic geometry for part (a) and Archimedes' method for part (b)).

   (a) $\int_{-4}^{0}(-0.5x+1)\,dx$
   (b) $\int_{-3}^{2}(-x^2-x+6)\,dx$

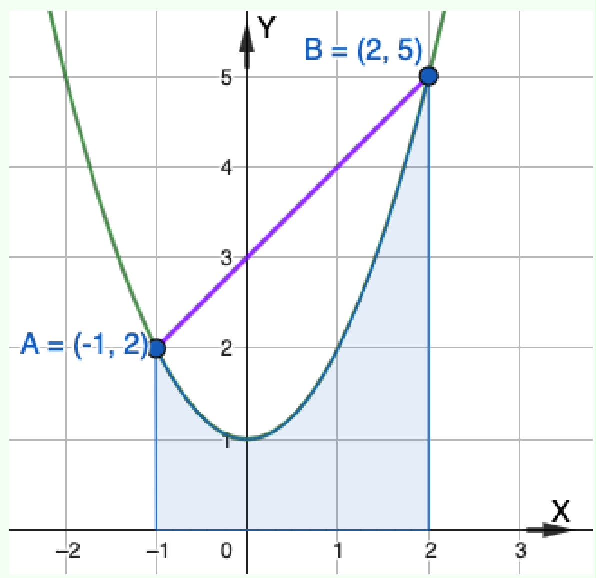
Archimedes' Method Extension

So far, all of the Archimedes' method questions have been designed so that the line AB is horizontal. This extension explores how to calculate the area under a parabola for a non-horizontal AB.

You will learn to calculate the area of this blue region, which is the area under the parabola and above the x-axis from $x = -1$ to $x = -2$. You will notice that it is very similar to Example 3 in the lesson, so the instructions for each step have been omitted.



1. The lower bound is $x = -1$. When $x = -1$, $y = (1)^2 + 1 = 2$, so point A is $(-1, 2)$. The upper bound is $x = 2$. When $x = 2$, $y = (2)^2 + 1 = 5$, so point B is $(2, 5)$.



2. The x-coordinate of point C is the midpoint of the bounds so $x = \frac{-1+2}{2} = \frac{1}{2}$.

   The y-coordinate is $y = (\frac{1}{2})^2 + 1 = \frac{5}{4}$ so point C is $(\frac{1}{2}, \frac{5}{4})$.

   Finding the area of this triangle requires addition work because the measurements of the base and height are unclear.

   We can find the area of rectangle FGBH and then we can subtract the three orange triangles to find the area of triangle ABC, A_{ABC}.

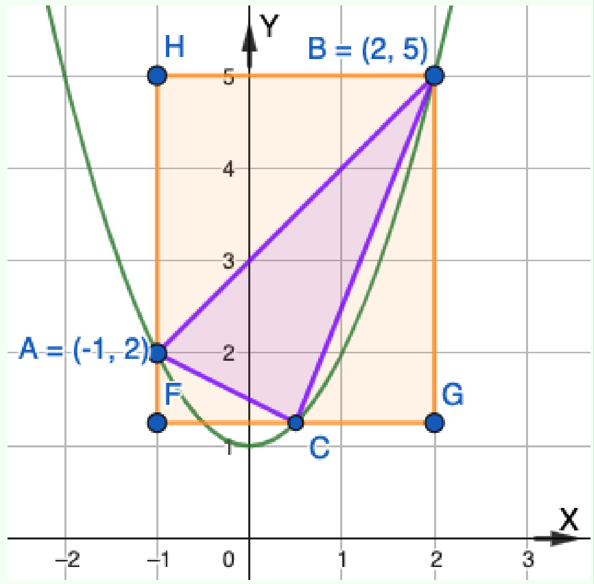
   $A_{FGBH} = 3 \times \frac{15}{4} = \frac{45}{4}$

   $A_{ABH} = \dfrac{3 \times 3}{2} = \frac{9}{2}$

   $A_{ACF} = \dfrac{\frac{3}{2} \times \frac{3}{4}}{2} = \frac{9}{16}$

   $A_{BCG} = \dfrac{\frac{3}{2} \times \frac{15}{4}}{2} = \frac{45}{16}$

   $A_{ABC} = \frac{45}{4} - \frac{9}{2} - \frac{9}{16} - \frac{45}{16} = \frac{54}{16} = \dfrac{27}{8}$.



3. Archimedes proved that the area enclosed by the line AB and the parabola (the red area) is four-thirds of the area of triangle ABC.

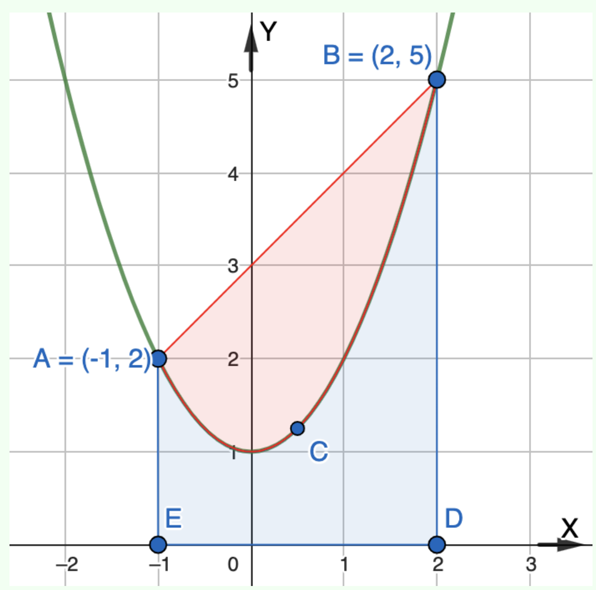
   $A_{red} = \frac{4}{3} \times \frac{27}{8} = \frac{9}{2}$

4. Finally, calculate the blue area by subtracting the red area from quadrilateral ABDE. Recall that the area of a trapezoid is the average of its parallel sides times its height.

   $A_{blue} = (\frac{2+5}{2} \times 3) - \frac{9}{2}$

   $A_{blue} = \frac{21}{2} - \frac{9}{2}$

   $A_{blue} = 6$



7. CHALLENGE PROBLEM 2

   Find the area under the parabola $y = x^2 - 2x + 1$ from the $x = -1$ to $x = 2$.

   HINT: If you did question 3(a), then you have already graphed this function!